

Factorization over the complex numbers

Factorize the followings up to linear complex number factors of the form $[z \pm (a + bi)]$, where a, b are real numbers. As an example, $z^2 + 1 = (z + i)(z - i)$

1. $z^4 - 5z^2 - 6$

$$z^4 - 5z^2 - 6 = (z^2 - 6)(z^2 + 1) = \underline{\underline{(z + \sqrt{6})(z - \sqrt{6})(z + i)(z - i)}}$$

2. $z^4 + 4$

$$\begin{aligned} z^4 + 4 &= (z^4 + 4z^2 + 4) - 4z^2 = (z^2 + 2)^2 - (2z)^2 = [z^2 + 2z + 2][z^2 - 2z + 2] \\ &= [(z^2 + 2z + 1) + 1][(z^2 - 2z + 1) + 1] = [(z + 1)^2 - i^2][(z - 1)^2 - i^2] \\ &= \underline{\underline{[z + (1 + i)][z + (1 - i)][z - (1 + i)][z - (1 - i)]}} \end{aligned}$$

3. $z^3 + z - 2$

$$\begin{aligned} z^3 + z - 2 &= (z - 1)(z^2 + z + 2) = (z - 1)\left(z - \frac{-1 + \sqrt{-7}}{2}\right)\left(z - \frac{-1 - \sqrt{-7}}{2}\right) \\ &= \underline{\underline{(z - 1)\left(z - \frac{-1 + \sqrt{7}i}{2}\right)\left(z - \frac{-1 - \sqrt{7}i}{2}\right)}} \end{aligned}$$

4. $z^5 + z^4 + z^3 + z^2 + z + 1$

Method 1

$$\begin{aligned} z^5 + z^4 + z^3 + z^2 + z + 1 &= (z + 1)(z^4 + z^2 + 1) = (z + 1)[(z^4 + 2z^2 + 1) - z^2] \\ &= (z + 1)[(z^2 + 1)^2 - z^2] = (z + 1)(z^2 - z + 1)(z^2 + z + 1) \\ &= \underline{\underline{(z + 1)\left[z - \frac{1 + \sqrt{3}i}{2}\right]\left[z - \frac{1 - \sqrt{3}i}{2}\right]\left[z - \frac{-1 + \sqrt{3}i}{2}\right]\left[z - \frac{-1 - \sqrt{3}i}{2}\right]}} \end{aligned}$$

Method 2

$$\begin{aligned} z^5 + z^4 + z^3 + z^2 + z + 1 &= z^3(z^2 + z + 1) + (z^2 + z + 1) = (z^3 + 1)(z^2 + z + 1) \\ &= (z + 1)(z^2 - z + 1)(z^2 + z + 1) = \underline{\underline{(z + 1)\left[z - \frac{1 + \sqrt{3}i}{2}\right]\left[z - \frac{1 - \sqrt{3}i}{2}\right]\left[z - \frac{-1 + \sqrt{3}i}{2}\right]\left[z - \frac{-1 - \sqrt{3}i}{2}\right]}} \end{aligned}$$

5. $z^4 - z^3 + 2z^2 - z + 1$

$$\begin{aligned} z^4 - z^3 + 2z^2 - z + 1 &= (z^4 - z^3 + z^2) + (z^2 - z + 1) = z^2(z^2 - z + 1) + (z^2 - z + 1) \\ &= (z^2 + 1)(z^2 - z + 1) \\ &= \underline{\underline{(z + i)(z - i) \left[z - \frac{1 + \sqrt{3}i}{2} \right] \left[z - \frac{1 - \sqrt{3}i}{2} \right]}} \end{aligned}$$

6. $z^3 + i$

$$\begin{aligned} z^3 + i &= z^3 - i^3 = (z - i)(z^2 + iz + i^2) = (z - i)(z^2 + iz - 1) \\ &= (z - i) \left(z - \frac{-i + \sqrt{i^2 - 4(1)(-1)}}{2} \right) \left(z - \frac{-i - \sqrt{i^2 - 4(1)(-1)}}{2} \right) \\ &= \underline{\underline{(z - i) \left(z - \frac{\sqrt{3} - i}{2} \right) \left(z - \frac{-\sqrt{3} - i}{2} \right)}} \end{aligned}$$

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